

# Anyonic Symmetries and Topological Defects in Abelian Topological Phases: an application to the *ADE* Classification

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We study symmetries and defects of a wide class of two dimensional Abelian topological phases characterized by Lie algebras. We formulate the symmetry group of all Abelian topological field theories. The symmetries relabel quasiparticles (or anyons) but leave exchange and braiding statistics unchanged. Within the class of *ADE* phases in particular, these anyonic symmetries have a natural origin from the Lie algebra. We classify one dimensional gapped phases along the interface between identical topological states according to symmetries. This classification also applies to gapped edges of a wide range of fractional quantum spin Hall(QSH) states. We show that the edge states of the *ADE* QSH systems can be gapped even in the presence of time reversal and charge conservation symmetry. We distinguish topological point defects according to anyonic symmetries and bound quasiparticles. Although in an Abelian system, they surprisingly exhibit non-Abelian fractional Majorana-like characteristics from their fusion behavior.

Topologically ordered phases with Abelian anyons are usually considered to be the simplest examples of topological order (TO), however recent exciting work has shown that the theory is still far from complete. Two notable developments related to our current work are: (i) the generation of semi-classical defects in Abelian topological phases that exhibit similar features to non-Abelian quasiparticles[1–12] and (ii) the bulk-boundary correspondence for topological phases with and without symmetry protection and the stability of the gapless edge theories [13–19].

Some aspects of these two lines of research can be unified by applying the concept of *anyonic* symmetry (AS). TO phases support an additional AS structure if the quasiparticle (QP) fusion and braiding are invariant under a set of anyon relabeling operations. This is a common feature in many topological states, such as the electric-magnetic duality in the Kitaev toric code [20] and the bilayer symmetry of an Abelian (*mmn*)-fractional quantum Hall (FQH) state [21–28]. An element of the AS group might, for example, switch a particular anyon-type between the two layers in bi-layer FQH states. The AS is not necessarily a symmetry of the quantum Hamiltonian, but rather a symmetry of the anyon content. For example, an AS could permute QP excitations with different energies. In general, a ground state in a closed system will not be invariant under an AS operation, and therefore the symmetry can be regarded as being *weakly* broken [1, 29]. However, unlike a classical symmetry-broken phase, the AS may not be associated with a physical quantity, and cannot be measured by a finite vacuum expectation value of any local observable.

In this work we construct a class of Abelian bosonic FQH states associated to elements of the *ADE* Cartan classification of Lie-algebras and show that they have AS. For these systems the AS can be used to create non-Abelian *twist defects* and topologically distinct gapped edge, or interface, phases. Remarkably we find an exact

mapping between the well-known *triality* symmetry of the Lie algebra  $so(8)$  and the AS of the associated topological state. In fact, we prove that the AS for these theories are exactly the symmetries of the Dynkin diagrams that represent the *ADE* Lie algebras. This is not only applicable to a 2D FQH state that carries an  $so(8)$  edge algebra, but also the spin liquid surface state of a three dimensional bosonic symmetry protected phase [30–33].

To begin, we need to introduce the well-known  $K$ -matrix formalism for Abelian TO states. An Abelian FQH state is described by an effective Chern-Simons topological field theory  $\mathcal{L} = \frac{1}{4\pi} K_{IJ} \alpha_I \wedge d\alpha_J$  in  $2+1$  dimensions, where  $\alpha_I$  is an  $r$ -component  $U(1)$  gauge field. The topological state is characterized by the symmetric, integral-valued  $K$ -matrix [23]. QP excitations of the theory are labeled as  $r$ -component vectors  $(\mathbf{a}, \mathbf{b} \dots)$  in an integer (anyon) lattice  $\Gamma^* = \mathbb{Z}^r$ . Vector addition corresponds to QP fusion  $\psi^{\mathbf{a}} \times \psi^{\mathbf{b}} = \psi^{\mathbf{a}+\mathbf{b}}$ . The spin (or exchange statistics) of a QP  $\psi^{\mathbf{a}}$  is given by  $\theta_{\mathbf{a}} = e^{\pi i \mathbf{a}^T K^{-1} \mathbf{a}}$ , and encircling a QP  $\psi^{\mathbf{a}}$  once around another QP  $\psi^{\mathbf{b}}$  gives the braiding phase  $\mathcal{D}S_{\mathbf{ab}} = e^{2\pi i \mathbf{a}^T K^{-1} \mathbf{b}}$ , for  $\mathcal{D} = \sqrt{|\det(K)|} \geq 1$ . The spin of the quasiparticles is often stated in terms of the  $T$  matrix,  $T_{\mathbf{ab}} = \delta_{\mathbf{a},\mathbf{b}} \theta_{\mathbf{a}}$ .

The QPs that occupy the sublattice  $\Gamma = K\mathbb{Z}^r \subseteq \Gamma^*$  are called *local* and only contribute trivial braiding phases with all other QPs. Intuitively they are the fundamental building blocks that are “fractionalized” to form the topological state; we assume all local particles are bosonic by requiring the diagonal entries of  $K$  be even. At zero temperature, the ground state is a Bose-Einstein condensate of local bosons with broken  $U(1)$  symmetry/number conservation. Physically, the boson condensate could describe an *anyonic superconductor* [34, 35] where local particles are Cooper pairs of electrons. QPs that differ by local bosons are indistinguishable and interchangeable up to the boson condensate vacuum. We can remove this redundancy by only labeling QPs with elements of the anyon quotient lattice  $\mathcal{A} = \Gamma^*/\Gamma = \mathbb{Z}^r/K\mathbb{Z}^r$ . QPs are

electromagnetically charged in the presence of the additional coupling term  $\frac{e^*}{2\pi} t_I A \wedge d\alpha_I$  where  $A$  is the external electromagnetic gauge field and  $e^*$  is the unit charge of the fundamental local boson. We will assume a symmetric coupling  $\mathbf{t} = (t_I) = (1, \dots, 1)$  which, for example, is the natural choice in multi-layer systems. The (fractional) charge of a QP  $\psi^{\mathbf{a}}$  is  $q_{\mathbf{a}} = e^* \mathbf{t}^T K^{-1} \mathbf{a}$ . Because of the boson condensate,  $q_{\mathbf{a}}$  is only defined modulo integral units of  $e^*$  at zero temperature.

We are interested in Abelian topological states which carry chiral Kac-Moody (KM) current algebras at level 1 along their edges. These include a range of FQH states under the Cartan  $ADE$  classification of simply-laced Lie algebras [36–38]. The set of  $A_r$  and  $D_r$  form infinite sequences while there are only three exceptional  $E_{r=6,7,8}$ . These Abelian states must contain a sector whose  $K$ -matrix is identical (up to stable equivalence) to the Cartan matrix of the Lie algebra (which has rank  $r$ ) [39]. Throughout, we will use two explicit examples to illustrate our results  $A_2 = su(3)$  and  $D_4 = so(8)$  which are described by

$$K_{su(3)} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad K_{so(8)} = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}. \quad (1)$$

The  $su(3)$  state has 3 QPs:  $1 = (0, 0)$ ,  $e = (-1, 1)$  and  $e^2 = (1, -1)$ , which form the anyon quotient lattice  $\mathcal{A}_{su(3)} = \mathbb{Z}_3$  with fusion  $e \times e = e^2$  and  $e \times e^2 = 1$  up to local bosons. The QPs have neutral electric charge, but have non-trivial spin  $\theta_e = \theta_{e^2} = e^{2\pi i/3}$  and braiding phases  $\sqrt{3}S_{ee} = \sqrt{3}S_{e^2e^2} = e^{-2\pi i/3}$  and  $\sqrt{3}S_{ee^2} = e^{2\pi i/3}$ . The anyon quotient lattice of  $su(3)$  is two dimensional and can be easily drawn. It is shown in Fig. 1, the white, red and blue circles refer to the quasiparticles  $1$ ,  $e$  and  $e^2$ .

The  $so(8)$  state has 4 QPs:  $1 = (0, 0, 0, 0)$ ,  $e = (0, -1, 0, 1)$ ,  $m = (0, -1, 1, 0)$  and  $\psi = (0, 0, -1, 1)$  forming the anyon quotient lattice  $\mathcal{A}_{so(8)} = \mathbb{Z}_2 \times \mathbb{Z}_2$  with fusion rules  $e^2 = m^2 = 1$  and  $\psi = e \times m$  up to local bosons. They have neutral electric charge, carry fermionic spin  $\theta_e = \theta_m = \theta_\psi = -1$  and braiding phases  $2S_{ee} = 2S_{mm} = 2S_{\psi\psi} = 1$  and  $2S_{em} = 2S_{m\psi} = 2S_{\psi e} = -1$ .

The anyon labels and fusion rules for general  $ADE$  states are listed in Table I [40]. In the Lie algebra language the lattice  $\Gamma = K\mathbb{Z}^r$  of local bosons is the *root* lattice, while the anyon lattice  $\Gamma^* = \mathbb{Z}^r$  is called the *weight* lattice [18] and is dual to  $\Gamma$  under the bilinear product  $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^T K^{-1} \mathbf{b}$ . We have omitted the  $E_8$  state with trivial topological order ( $\mathcal{D} = 1$ ) [41]. We note in passing that there is an eightfold periodicity in the  $D_r$  series with rank  $r \geq 3$  such that the  $D_r$  state is stably equivalent [18] to the  $D_{r+8}$  theory up to an additional  $E_8$  state [39], and both theories have identical anyon fusion and braiding content. Taken together with the non-simply laced

$B_r$  series (which are non-Abelian at level-1 and will be discussed elsewhere [39]), they form a class of topological states with sixteenfold periodicity which matches the structure found in Refs. 1, 42, and 43. Also, from the braiding phase and spin of the quasiparticles we expect that  $K_{E_7} \oplus \sigma_x$  is stably equivalent to  $(-K_{su(2)}) \oplus E_8$  and  $K_{E_6} \oplus \sigma_x \oplus \sigma_x$  is stably equivalent to  $(-K_{su(3)}) \oplus E_8$ .

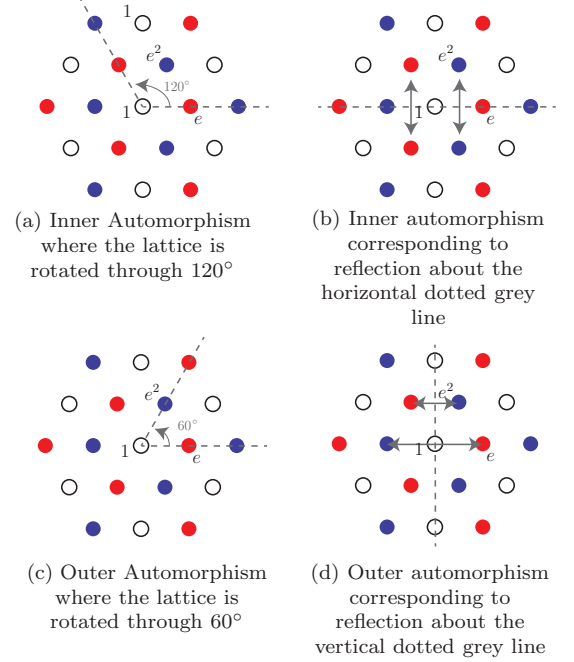


FIG. 1. Anyon lattice of  $su(3)$  with inner and outer automorphisms. The white, red and blue circles refer to the distinct quasiparticles  $1$ ,  $e$  and  $e^2$  respectively in the anyon quotient lattice  $\mathbb{Z}^2/K\mathbb{Z}^2$ . (a) and (b) are examples of inner automorphisms where anyon labels (colors of the circles) are preserved whereas (c) and (d) the outer automorphisms exchange  $e$  and  $e^2$  (blue and red circles).

	Anyon fusion lattice $\mathcal{A} = \mathbb{Z}^r / K\mathbb{Z}^r$	Anyon labels
$A_r$	$\mathbb{Z}_{r+1}$	$1 = e^{r+1}, e, \dots, e^r$
$D_{2n}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$1, e, m, \psi = e \times m$
$D_{2n+1}$	$\mathbb{Z}_4$	$1 = e^4, e, e^2, e^3$
$E_6$	$\mathbb{Z}_3$	$1 = e^3, e, e^2$
$E_7$	$\mathbb{Z}_2$	$1 = e^2, e$

TABLE I. Quasiparticle labels of the  $A_r = su(r+1)$ ,  $D_r = so(2r)$  and  $E_{6,7}$  Abelian topological states at level 1.

We will now construct the anyonic symmetry groups for the  $ADE$  states and discuss applications of this result in the context of symmetry enhanced topological phases and semi-classical twist defects. One requirement of an AS is that its operation commutes with the modular  $S$  and  $T$  transformations of TO states on a torus. For

Abelian theories in the  $K$ -matrix formalism, a unitary anyon relabeling symmetry can be represented by a unimodular (integral entries, unit determinant) matrix  $M$  that leaves the  $K$ -matrix invariant under  $MKM^T$ . This forms a group of automorphisms

$$\text{Aut}(K) = \{M \in GL(r; \mathbb{Z}) : MKM^T = K\}. \quad (2)$$

Since exchange and braiding are completely determined by the  $K$ -matrix, the modular transformations are unchanged under the anyon relabeling,

$$S_{M\mathbf{a}M\mathbf{b}} = S_{\mathbf{a}\mathbf{b}}, \quad T_{M\mathbf{a}M\mathbf{b}} = T_{\mathbf{a}\mathbf{b}} \quad (3)$$

and the fusion rules remain unaltered as a direct consequence of the linearity of  $M$  or the Verlinde formula [44] in general.

Out of the full group of automorphisms there are certain trivial symmetry operations  $M_0$  that only rearrange *local* particles without changing the QP types. That is, these operations do not change the anyon equivalence classes  $[\mathbf{a}] = \mathbf{a} + K\mathbb{Z}^r \in \mathcal{A}$  since they rotate the anyon lattice vector up to a local particle in  $\Gamma = K\mathbb{Z}^r$ .  $M_0$  forms a normal subgroup called the *inner automorphisms*

$$\text{Inner}(K) = \{M_0 \in \text{Aut}(K) : [M_0\mathbf{a}] = [\mathbf{a}]\}. \quad (4)$$

To construct the relevant AS group we must remove this redundancy of trivial symmetry operations by quotienting, to generate the group known as the *outer automorphisms*

$$\text{Outer}(K) = \frac{\text{Aut}(K)}{\text{Inner}(K)}. \quad (5)$$

Thus,  $\text{Outer}(K)$  is the AS group of the Abelian topological phase characterized by  $K$ . If the topological state is strictly charge conserving, a charge compatible AS element must keep the charge vector  $M\mathbf{t} = \mathbf{t}$  fixed which will ensure the charge of a QP is unchanged,  $q_{M\mathbf{a}} = q_{\mathbf{a}}$ . For a  $U(1)$ -breaking bosonic state at zero temperature, the charge compatibility condition can be relaxed modulo the image of  $K$  so that the fractional charge is preserved by the symmetry only up to units of  $e^*$  through the addition of local particles. Imposing these charge conservation conditions will further restrict the group of automorphisms in (2).

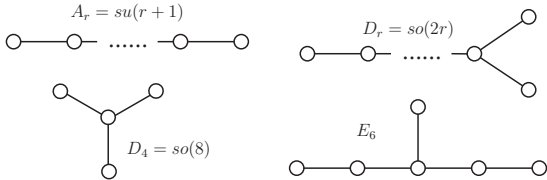


FIG. 2. Mirror symmetry of Dynkin Diagrams of  $A_r, D_r, E_6$  and  $S_3 = Dih_3$  symmetry of  $D_4$ .

	$\text{Outer}(K)$	anyonic symmetry action
$A_r, D_{2n+1}, E_6$	$\mathbb{Z}_2$	$\sigma : e \leftrightarrow e^{-1}$
$D_{2n}, n \neq 2$	$\mathbb{Z}_2$	$\sigma : e \leftrightarrow m$
$D_4$	$S_3 = Dih_3$	$\sigma_\psi : e \leftrightarrow m$ $\rho : e \rightarrow m \rightarrow \psi \rightarrow e$

TABLE II. Symmetry action  $\text{Outer}(K)$  on anyon labels.

Remarkably, in an Abelian state in the  $ADE$  classification,  $\text{Outer}(K)$  is exactly the group of outer automorphisms of the simply-laced Lie algebra, and coincides with the symmetry group of the Dynkin diagram [45, 46] (see Fig. 2). The explicit AS actions are listed in Table II. For  $A_2 = su(3)$  the Dynkin diagram has a  $\mathbb{Z}_2$  “reflection” AS which is represented by the Pauli matrix  $M_\sigma = \sigma_x$  that acts on the rank two anyon lattice vectors and simply interchanges the QPs  $e$  and  $e^2$  while leaving the vacuum fixed. Examples of inner and outer automorphisms are shown in the Fig. 1.

The only  $ADE$  state with more than just a  $\mathbb{Z}_2$  AS group is the  $D_4 = so(8)$  state which has a triality symmetry. The AS group is  $S_3 = Dih_3$  which is the permutation group of three elements generated by “reflection” and threefold “rotation” in its Dynkin diagram. This group contains a total of six elements, and the generators are represented by

$$M_{\sigma_\psi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad M_\rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (6)$$

which act on four dimensional anyon lattice vectors.  $M_{\sigma_\psi}$  interchanges  $e \leftrightarrow m$  but fixes  $\psi$  up to local boson.  $M_\rho$  rotates  $e \rightarrow m \rightarrow \psi \rightarrow e$  which is an example of a threefold symmetry operation. All  $ADE$  symmetry operations can be chosen to strictly preserve  $U(1)$  symmetry and leave the charge vector unchanged so we have no further restrictions.

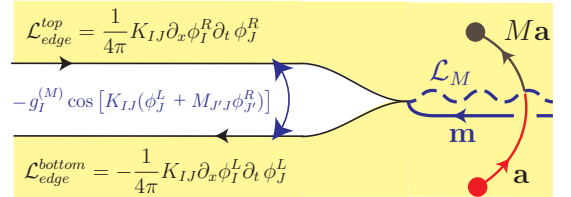


FIG. 3. Local boson tunneling in (8) gives a gapped interface phase  $\mathcal{L}_M$  represented by a branch cut (dashed wavy blue line) and a parallel quasiparticle string (solid blue line). Passing anyon changes type  $\mathbf{a} \rightarrow M\mathbf{a}$  and accumulate a crossing phase  $\mathcal{DS}_{\mathbf{a}\mathbf{m}}$  in (10).

Anyonic symmetry can give rise to topologically distinct gapped phases [17] along quasi-one dimensional interfaces with identical, but oppositely propagating, edge

modes (see Fig. 3). So, to consider some applications of our results we need to consider the gapless edge theories of the TO states. The gapless edge is characterized by the  $(1+1)$ -d bosonic Lagrangian density  $\mathcal{L}_{edge} = \frac{1}{4\pi} K_{IJ} \partial_x \phi_I \partial_t \phi_J$ , where the  $K$ -matrix is identical to that in the bulk, and QPs are expressed as vertex operators  $\psi^{\mathbf{a}} = e^{i\mathbf{a} \cdot \phi}$ . In the weak coupling limit, the chiral gapless edge modes along opposite sides of the interface are described by the boson Lagrangian density

$$\mathcal{L}_{edge}^{top} + \mathcal{L}_{edge}^{bottom} = \frac{1}{4\pi} K_{IJ}^{\sigma\sigma'} \partial_x \phi_I^\sigma \partial_t \phi_J^{\sigma'} + \frac{1}{2\pi} t_I^\sigma \epsilon^{\mu\nu} \partial_\mu \phi_I^\sigma A_\nu \quad (7)$$

where  $\sigma = 0, 1 = R, L$  labels right and left moving modes,  $\phi_I^R$  ( $\phi_I^L$ ) are the boson fields living along the top (bottom) edge, and  $K_{IJ}^{\sigma\sigma'} = (-1)^\sigma \delta^{\sigma\sigma'} K_{IJ}$ . We assume that  $\mathbf{t}^\sigma = \mathbf{t}^L = \mathbf{t}^R = \mathbf{t}$ . Corresponding to each quasiparticle vector  $\mathbf{a}$  in the bulk, the vertex operator on the right(left) edge is  $\psi_R^{\mathbf{a}} = e^{i\mathbf{a} \cdot \phi^R}$  ( $\psi_L^{\mathbf{a}} = e^{-i\mathbf{a} \cdot \phi^L}$ ). This convention ensures that the charge of both  $\psi_R^{\mathbf{a}}$  and  $\psi_L^{\mathbf{a}}$  is  $e^* \mathbf{a}^T K^{-1} \mathbf{t}$ .

Interestingly, given any element  $M$  of the AS group, the interface can be gapped by the set of backscattering terms

$$\delta \mathcal{L}_M = - \sum_I g_I^{(M)} \cos [K_{IJ} (\phi_J^L + M_{J'J} \phi_{J'}^R)] \quad (8)$$

where repeated indices  $J$  and  $J'$  are summed over. This describes tunneling between the local boson  $e^{iK_{IJ} M_{J'J} \phi_{J'}^R}$  on the top edge and  $e^{-iK_{IJ} \phi_J^L}$  on the bottom. We assume that the representative matrix of  $M$  is charge conserving so that the tunneling term preserves boson number, however, this condition can be relaxed if the charge  $U(1)$  is broken.

In the strong-coupling limit the terms in Eq. (8) lead to a gapped interface phase associated to the symmetry  $M$  which we denote by  $\mathcal{L}_M$ . The backscattering terms simultaneously pin the boson vacuum expectation values

$$\langle \phi_I^L + M_{JI} \phi_J^R \rangle = 2\pi (K^{-1})_{IJ} m_J, \quad \text{for } \mathbf{m} \in \mathbb{Z}^r \quad (9)$$

as the operators for each value of  $I$  mutually commute. This pinning effectively condenses the local bosonic QP pairs

$$\langle (\psi_R^{M\mathbf{a}})^\dagger \psi_L^{\mathbf{a}} \rangle = \langle e^{-i(\mathbf{a} \cdot \phi^L + (M\mathbf{a}) \cdot \phi^R)} \rangle = e^{-2\pi i \mathbf{a}^T K^{-1} \mathbf{m}} \quad (10)$$

along the interface. The gapped interface state is then characterized by this QP pair condensation, and can be diagrammatically represented by a branch cut associated to  $M$  that is decorated with a parallel QP string  $\mathbf{m}$  localized near the cut (see Fig. 3). The branch cut itself will change the anyon type of a passing QP from  $\mathbf{a} \rightarrow M\mathbf{a}$ , while the attached  $\mathbf{m}$  string contributes the additional

$U(1)$  crossing phase required from Eq. (10). Such QPs localized at defects have also been studied by [17, 47–50]. Two symmetry matrices  $M$  and  $M'$  correspond the same gapped interface phase if the backscattering terms pin and condense the same set of bosonic QP pairs, i.e., the gapped edge phases are identical if  $M\mathbf{a} = M'\mathbf{a}$  modulo  $K\mathbb{Z}^r$ . Gapped phases are therefore in one-to-one correspondence to the group  $\text{Outer}(K)$  of anyonic relabeling symmetries defined in (5). We further note that, although we focus on charge conserving edge tunneling terms in this work, our formalism also applies to gapping terms which describes superconducting pairing. This includes terms like

$$\begin{aligned} \delta \mathcal{L}_M &= - \sum_I g_I^{(M)} e^{iK_{IJ} M_{J'J} \phi_J^R} e^{-iK_{IJ} \phi_J^L} + h.c. \\ &= - \sum_I g_I^{(M)} \cos [K_{IJ} (\phi_J^L - M_{J'J} \phi_{J'}^R)] \end{aligned}$$

which condense local bosonic pairs  $\psi_R^{M\mathbf{a}} \psi_L^{\mathbf{a}}$  on the edge.

Now, with this setup we are equipped to consider two interesting problems: the gapped edge phases of *ADE* fractional quantum spin Hall phases and *ADE* twist defects. To approach the first problem we note that the interface (7) can be regarded as a single-edge of a doubled system with  $K$ -matrix  $K^{\sigma\sigma'} = K \oplus (-K)$  if the topological states on the two sides of the interface are folded on top of each other [16, 17]. This represents a fractional quantum spin Hall state of bosons [51, 52] with time reversal (TR) matrix  $T_{IJ}^{\sigma\sigma'} = (\sigma_x)^{\sigma\sigma'} \delta_{IJ}$  acting on the spin-momentum locked  $\sigma = \uparrow, \downarrow = R, L$  degree of freedom. The TR operator  $\mathcal{T}$  is anti-unitary and acts according to

$$\mathcal{T}^{-1} \phi_I^\sigma \mathcal{T} = T_{IJ}^{\sigma\sigma'} \phi_J^{\sigma'} + \pi (K^{-1})_{IJ}^{\sigma\sigma'} \chi_J^{\sigma'} \quad (11)$$

for some TR vector  $\chi = (\chi_\uparrow, \chi_\downarrow) \in \mathbb{R}^{2r}$  [52]. For the doubled ADE systems we show in the Supplement that  $\chi$  can be set to 0 and hence,  $\mathcal{T}^{-1} \phi_I^{L/R} \mathcal{T} = \phi_I^{R/L}$ . Thus,  $\mathcal{T}^{-1} \psi_R^{\mathbf{a}} \mathcal{T} = \psi_L^{\mathbf{a}}$ . Indeed since  $\psi_R^{\mathbf{a}}$  and  $\psi_L^{\mathbf{a}}$  are time-reversed partners, their spin and S matrices obey  $\theta_R^{\mathbf{a}} = (\theta_L^{\mathbf{a}})^* = e^{i\pi \mathbf{a}^T K^{-1} \mathbf{b}}$  and  $S_R^{\mathbf{a}\mathbf{b}} = (S_L^{\mathbf{a}\mathbf{b}})^* = e^{i\pi \mathbf{a}^T K^{-1} \mathbf{b}}$ .

Generically, since the edge is non-chiral, one can destabilize the edge and open a gap via, for example, condensing bosons on the edge. However, we are not only interested if a gap can form, but what symmetries the resulting gap state breaks or preserves, e.g., some gapped phases may break time-reversal and some may preserve time-reversal. For our problem, the edge condensate is formed from the QP pairs  $(\psi_\uparrow^{M\mathbf{a}})^\dagger \psi_\downarrow^{\mathbf{a}}$ , and is a maximal collection of mutually local bosons, known as a *Lagrangian subgroup* [15] in the TR symmetric doubled anyon lattice system  $\mathcal{A}_\uparrow \otimes \mathcal{A}_\downarrow$ .

Let us look at a few examples before making a general statement. To simplify notation let  $x_{R/L}$  stand for  $\psi_{R/L}^{\mathbf{x}}$  and  $\bar{x}_{R/L} \equiv (\psi_{R/L}^{\mathbf{x}})^\dagger$ . The  $A_2 = su(3)$



state has  $\text{AS Outer}(K_{su(3)}) = \mathbb{Z}_2$  generated by 1 and  $\sigma$ . These generators correspond to the gapped interface/edge phases  $\mathcal{L}_1$  and  $\mathcal{L}_\sigma$  with QP pair condensates  $\{1_L \bar{1}_R, e_L \bar{e}_R, e_L^2 \bar{e}_R^2\}$  and  $\{1_L \bar{1}_R, e_L \bar{e}_R^2, e_L^2 \bar{e}_R\}$  respectively. To test if these phases break TR explicitly we calculate  $\mathcal{T}^{-1} \mathcal{L}_1 \mathcal{T} = \{1_R \bar{1}_L, e_R \bar{e}_L, e_R^2 \bar{e}_L^2\}$  and  $\mathcal{T}^{-1} \mathcal{L}_\sigma \mathcal{T} = \{1_R \bar{1}_L, e_R \bar{e}_L^2, e_R^2 \bar{e}_L\}$ . To compare with  $\mathcal{L}_1$  and  $\mathcal{L}_\sigma$  we take the Hermitian conjugate, and we see that  $\mathcal{T}^{-1} \mathcal{L}_1 \mathcal{T} = (\mathcal{L}_1)^\dagger$  and  $\mathcal{T}^{-1} \mathcal{L}_\sigma \mathcal{T} = (\mathcal{L}_\sigma)^\dagger$ . While this might initially seem problematic for TR preservation, we note that  $\mathcal{L}$  and  $\mathcal{L}^\dagger$  actually represent the same set of condensed bosons. We see by using equations (9) and (10) that

$$\langle (\psi_R^{M\mathbf{a}})^\dagger \psi_L^{\mathbf{a}} \rangle \neq 0 \implies \langle \psi_R^{M\mathbf{a}} (\psi_L^{\mathbf{a}})^\dagger \rangle \neq 0.$$

Explicitly, we can see this by taking the Hermitian conjugate of equation (10)

$$\langle [(\psi_R^{M\mathbf{a}})^\dagger \psi_L^{\mathbf{a}}]^\dagger \rangle = \langle \psi_R^{M\mathbf{a}} (\psi_L^{\mathbf{a}})^\dagger \rangle = e^{2\pi i \mathbf{a}^T K^{-1} \mathbf{m}}.$$

Hence, we see that the time reversed (i.e., Hermitian conjugate) phase condenses the same bosons as the original one, i.e. it does not break time reversal *explicitly*.

The  $D_4 = so(8)$  state has six symmetry operators in  $\text{Outer}(K_{so(8)}) = S_3$ , and correspond six gapped phases. For example the trivial one  $\mathcal{L}_1$  condenses  $\{1_L \bar{1}_R, e_L \bar{e}_R, m_L \bar{m}_R, \psi_L \bar{\psi}_R\}$ , the twofold one  $\mathcal{L}_{\sigma_\psi}$  condenses  $\{1_L \bar{1}_R, e_L \bar{m}_R, m_L \bar{e}_R, \psi_L \bar{\psi}_R\}$ , and the threefold one  $\mathcal{L}_\rho$  condenses  $\{1_L \bar{1}_R, e_L \bar{m}_R, m_L \bar{\psi}_R, \psi_L \bar{e}_R\}$ . The Lagrangian subgroup for  $\mathcal{L}_{\sigma_\psi}$  preserves TR (analogous to  $\mathcal{L}_\sigma$  for  $su(3)$ ) while that for  $\mathcal{L}_\rho$  breaks it upon adding the gapping terms. We consider the example for  $\mathcal{L}_\rho$  explicitly,  $\mathcal{T}^{-1} \mathcal{L}_\rho \mathcal{T} = \{1_R \bar{1}_L, e_R \bar{m}_L, m_R \bar{\psi}_L, \psi_R \bar{e}_L\}$ . Taking hermitian conjugate we see that  $\{1_L \bar{1}_R, m_L \bar{e}_R, \psi_L \bar{m}_R, e_L \bar{\psi}_R\} = \mathcal{L}_{\rho^{-1}}$ . Hence the threefold defect is certainly not time reversal invariant, because the time-reversed phase condenses a different set of bosons; this phase breaks time reversal *explicitly*.

In the Supplement we prove that the gapped edge phase  $\mathcal{L}_M$  does not break TR explicitly if the Lagrangian subgroup is TR invariant, i.e., the TR of the gapped edge describes the same phase, which is the case if and only if  $M^2 = 1$  (up to inner automorphisms). Microscopically this arises from the fact that if  $M^2 = 1$  then the sine-Gordon gapping terms in (8) can be made TR symmetric by adding time-reversed counterparts *that pin the same set of QP pairs*. This includes all symmetries in the ADE states except the threefold symmetry  $\rho$  of  $so(8)$ . One cannot write down a time-reversal invariant Lagrangian that is gapped by the three-fold symmetry phase.

There is a further complication in that, even though the Lagrangian is TR invariant, the ground state condensate itself can break TR spontaneously, because even

though the gapped interface may pin the same QPs as its TR partner, the expectation value of the condensate in the time reversed phase may be different than that in the original phase. In order to prevent the spontaneous breaking of TR symmetry we must constrain the condensate phase as follows. First, let us specialize to gapped edges where  $M^2 = 1$  so that TR is not broken explicitly. Time reversal (11) operates on the QP condensate  $(\psi_R^{M\mathbf{a}})^\dagger \psi_L^{\mathbf{a}}$  along a quantum spin Hall edge by

$$\begin{aligned} \mathcal{T}^{-1} [(\psi_R^{M\mathbf{a}})^\dagger \psi_L^{\mathbf{a}}] \mathcal{T} &= \mathcal{T}^{-1} [e^{-i((M\mathbf{a}) \cdot \phi^R + \mathbf{a} \cdot \phi^L)}] \mathcal{T} \\ &= e^{i((M\mathbf{a}) \cdot \phi^L + \mathbf{a} \cdot \phi^R)} = e^{i((MM\mathbf{a}) \cdot \phi^R + M\mathbf{a} \cdot \phi^L)}. \end{aligned}$$

Using Eq. (9) we find

$$\langle \mathcal{T}^{-1} [(\psi_R^{M\mathbf{a}})^\dagger \psi_L^{\mathbf{a}}] \mathcal{T} \rangle = e^{2\pi i \mathbf{a}^T K^{-1} (M\mathbf{m})},$$

which should be compared to  $\langle (\psi_R^{M\mathbf{a}})^\dagger \psi_L^{\mathbf{a}} \rangle = e^{-2\pi i \mathbf{a}^T K^{-1} \mathbf{m}}$  from Eq. (10). Thus, to preserve time-reversal we must have  $\mathbf{m}$  satisfy

$$\mathbf{m} \equiv -M\mathbf{m} \pmod{K} \quad (12)$$

otherwise the ground state would break TR *spontaneously*. In all the ADE cases, we have examples of  $M$  which obey  $M^2 = 1$  and conserve charge  $M\mathbf{t} = \mathbf{t}$  (Eqs. (6) and (29)). We can also satisfy Eq. (12) with the case when  $m = 0$ . *Thus in all the ADE FQSH cases, we can gap out the edge even if we demand TRI and charge conservation.*

In fact using criterion in [52], one can shown that whenever  $\chi = 0$  the edge can be gapped without breaking TR or charge conservation. In particular these gapped edges represent twofold defects  $M^2 = 1$ . However, there exist gapped edges which do break time reversal symmetry explicitly while conserving charge, e.g., the threefold defect  $\rho$  for  $so(8)$ .

In addition to gapping symmetry protected edge states, we can apply our results to discuss twist defects associated with an AS element  $M$ , i.e., a topological point defect that changes the anyon type of QPs that travel around it according to  $\mathbf{a} \rightarrow M\mathbf{a}$  (see Fig. 4b). As discussed above, twist defects can essentially be considered as a domain wall sandwiched between two distinct gapped interface phases [17]. They can also appear attached to dislocations or disclinations (see Fig. 4a) where non-trivial boundary conditions (i.e. local boson tunneling) are applied at the extra inserted half-layer or wedge respectively. The classical defect interpretations are particularly relevant when the TO intertwines with (liquid) crystalline order where a broken discrete spatial symmetry matches an AS [11, 20, 53, 54].

According to Eq. (10) a twist defect  $\mu$  can bind a QP  $\mathbf{m}$  (see Fig. 4a,c). The defect-QP composite is summarized by a defect *species* label  $\lambda$  where  $\mu_\lambda \equiv \mu_0 \times \mathbf{m}$ ,

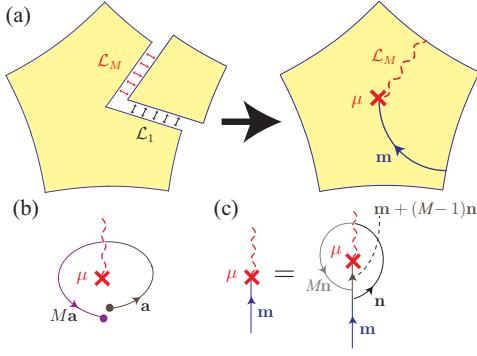


FIG. 4. (a) Twist defect  $\mu$  (red cross) as a disclination at the end of a gapped interface  $\mathcal{L}_M$  (dashed curvy line) and dressed with a quasiparticle string  $\mathbf{m}$  (solid line). (b) Defect action on a circling quasiparticle  $\mathbf{a} \rightarrow M\mathbf{a}$ . (c) Redundancy in defect-quasiparticle fusion.

where  $\mu_0$  is a *bare* defect with no attached QP, and  $\times$  means fusion. Because of the nature of the twist defect there is an important consistency constraint that must be satisfied when determining the possible attached QP types, i.e., there are redundancies in defect-quasiparticle fusion. For instance, as shown in Fig. 4c, when the QP  $\mathbf{m}$  is fusing with the defect  $\mu$  it can emit a QP  $\mathbf{n}$  that travels around the twist defect and is re-absorbed after (possibly) changing its anyon type. If the twist defect does not transform  $\mathbf{n}$  then it is simply re-absorbed without issue, however if  $\mathbf{n}$  is transformed then a physical consistency condition must be satisfied that leads to a redundancy in the possible defect species types. The constraint that must be satisfied is

$$\mu \times \mathbf{m} = \mu \times (\mathbf{m} + (M-1)\mathbf{n}) \quad (13)$$

and the defect species labels are thus classified by equivalence classes  $\lambda = \llbracket \mathbf{m} \rrbracket \equiv [m]$  modulo  $(M-1)\mathcal{A}$  [12]. A heuristic way to understand this constraint is that if the emitted quasi-particle  $\mathbf{n}$  is changed to  $M\mathbf{n}$  by the twist defect then the defect itself must be able to absorb the difference between  $\mathbf{n}$  and  $M\mathbf{n}$  to give a physically consistent result. As we will see, this means that the defects themselves can be thought of heuristically to “contain” internal structure that increases their quantum dimension and allow them to compensate for this emission-reabsorption process.

Take the the  $A_r = su(r+1)$  state for example. By solving the redundancy constraint we can see that the non-redundant species label only represents the parity of QP  $e^p$  bound to a twofold defect  $\sigma$ . However, the parity is only well defined only when the  $A_r$  has well defined even and odd QP sectors, i.e. when there are even number of QPs when  $r$  is odd. Thus  $A_r$ , when  $r$  is even, only has one type of defect species (equivalent to the bare defect), and when  $r$  is odd there are two types, an even and odd defect  $\sigma_0$  and  $\sigma_1$ . For the  $D_4 = so(8)$  state, while each twofold defect  $\sigma_e, \sigma_m, \sigma_\psi$  comes with two distinct species labels,

there is no non-trivial species label for threefold defects  $\rho, \bar{\rho}$  as their symmetries mix even and odd QP sectors. We notice that in the cases when charge is fractionalized (i.e. the  $A_r$  states for  $r$  odd and  $D_r$  states for  $r \equiv 2, 3 \pmod{4}$ ), the  $\mathbb{Z}_2$  species label also counts the fractional electric charge in units of  $e^*/2$  carried by the defect (c.f. Ref. 12).

defects and species		$d$
$A_{2n}$	$\sigma = \sigma \times e$	$\sqrt{2n+1}$
$A_{2n+1}$	$\sigma_0 = \sigma_0 \times e^2, \sigma_1 = \sigma_0 \times e$	$\sqrt{n+1}$
$D_{2n}$	$\sigma_0 = \sigma_0 \times \psi, \sigma_1 = \sigma_0 \times \{e, m\}$	$\sqrt{2}$
$D_{2n+1}$	$\sigma_0 = \sigma_0 \times e^2, \sigma_1 = \sigma_0 \times e$	$\sqrt{2}$
$D_4$	$\rho, \bar{\rho}$	2
$E_6$	$(\sigma_\psi)_0 = (\sigma_\psi)_0 \times \psi, (\sigma_\psi)_1 = (\sigma_\psi)_0 \times e$	$\sqrt{2}$
	$(\sigma_e)_0 = (\sigma_e)_0 \times e, (\sigma_e)_1 = (\sigma_e)_0 \times m$	$\sqrt{2}$
	$(\sigma_m)_0 = (\sigma_m)_0 \times m, (\sigma_m)_1 = (\sigma_m)_0 \times \psi$	$\sqrt{2}$
	$\sigma = \sigma \times e$	$\sqrt{3}$

TABLE III. Defects and species for the  $A-D-E$  states, and their quantum dimensions  $d$ .

Defect species can undergo a *mutation* by absorbing or releasing a quasiparticle. This can be microscopically controlled, for example, by adding phase parameters in the sine-Gordon terms in (8), and letting them wind adiabatically by multiple of  $2\pi$ . The mutation process is summarized by the defect-quasiparticle fusion

$$\sigma_{\lambda+p} = \sigma_\lambda \times e^p \quad (14)$$

where  $\lambda \in \mathbb{Z}_2 = \{0, 1\}$  for the  $A_{2n+1}$  and  $D_{2n+1}$  states. For the  $D_{2n}$  state,

$$\sigma_{\lambda+1} = \sigma_\lambda \times e = \sigma_\lambda \times m, \quad \sigma_\lambda = \sigma_\lambda \times \psi \quad (15)$$

Or for the  $D_4$  state,  $\sigma_\psi$  obeys (15) and the other two twofold defects  $\sigma_e$  and  $\sigma_m$  follow (15) up to cyclic permutation of quasiparticles.

The anti-partner of a defect  $\mu$  with symmetry  $M$  is a defect  $\bar{\mu}$  with symmetry  $M^{-1}$  and reciprocal species label  $\bar{\lambda} = -\lambda$ . As  $MM^{-1} = 1$ , a QP will not change type when dragged around the  $\mu \times \bar{\mu}$  pair. This gives a Wilson measurement of the overall QP type associated to the defect *pair*, known as an Abelian fusion channel. These channels are restricted only by defect species since QP parity (or half  $e^*$  charge) is a conserved property. Fusions of twofold defect pairs in the  $ADE$  states are summarized in Table IV. They fix the quantum dimensions of defects (shown in Table III) by identifying the total dimensions on both sides of the fusion equations. We see that fusion of two defects can give rise to a large number of types of QPs, and these possible internal states of two defects are precisely the same structure that allows them to compensate for the QP attachment emission-reabsorption constraint discussed above.

	Fusion rules
$A_{2n}$	$\sigma \times \sigma = 1 + e + \dots + e^{2n}$
$A_{2n+1}$	$\sigma_0 \times \sigma_0 = \sigma_1 \times \sigma_1 = 1 + e^2 + \dots + e^{2n}$ $\sigma_0 \times \sigma_1 = e + e^3 + \dots + e^{2n+1}$
$D_{2n}$	$\sigma_0 \times \sigma_0 = \sigma_1 \times \sigma_1 = 1 + \psi$ $\sigma_0 \times \sigma_1 = e + m$
$D_{2n+1}$	$\sigma_0 \times \sigma_0 = \sigma_1 \times \sigma_1 = 1 + e^2$ $\sigma_0 \times \sigma_1 = e + e^3$
$E_6$	$\sigma \times \sigma = 1 + e + e^2$

TABLE IV. Defect pair fusion in the  $A - D - E$  states.

There is more structure in the  $D_4 = so(8)$  state due to the  $S_3$  triality symmetry.  $(\sigma_\psi)_\lambda \times (\sigma_\psi)_{\lambda'}$  obeys the same fusion rules as the  $D_{2n}$  states in Table IV, while the two other twofold defects  $\sigma_e, \sigma_m$  satisfy similar rules up to a cyclic permutation of quasiparticles. The threefold defect  $\rho$  annihilates its anti-partner  $\bar{\rho}$  and gives

$$\rho \times \bar{\rho} = \bar{\rho} \times \rho = 1 + e + m + \psi. \quad (16)$$

This implies the quantum dimension  $d_\rho = d_{\bar{\rho}} = 2$  and degenerate fusion of the pair

$$\rho \times \rho = 2\bar{\rho}, \quad \bar{\rho} \times \bar{\rho} = 2\rho. \quad (17)$$

The non-Abelian symmetry group  $S_3$  results in non-commutative fusion rules

$$\sigma_m \times \sigma_e = \rho, \quad \sigma_e \times \sigma_m = \bar{\rho} \quad (18)$$

$$\sigma_e \times \rho = \rho \times \sigma_m = (\sigma_\psi)_0 + (\sigma_\psi)_1. \quad (19)$$

All other fusion rules can be written down by cyclic permutation of the quasiparticle labels  $e, m, \psi$ .

While we have not considered it here, we note that consideration of anyonic symmetries up to stable equivalence leads to an even richer structure of symmetries. So far we have considered anyonic symmetries satisfying Eq. (2). These symmetries preserve the  $K$  matrix and the scaling dimensions  $h_{\mathbf{a}}$  of the quasiparticles  $h_{\mathbf{a}} = \frac{1}{2}\mathbf{a}^T K^{-1}\mathbf{a}$ . In other words, the conformal structure on the edge is respected. However, we might relax this constraint and only require keeping the  $T$  matrix invariant where  $T_{\mathbf{a},\mathbf{a}} = e^{2\pi i h_{\mathbf{a}}}$ , then  $h_{\mathbf{a}}$  will only be preserved mod 1. This will lead to more anyonic symmetries as considered in Ref. 55. For example, in the case of  $su(12)$  the symmetries which preserve scaling dimensions lead to QP transformations generated by  $e \rightarrow e^{-1} = e^{11}$  (Table II).  $h_e = h_{e^{-1}} = \frac{11}{24}$  (see Supplement). However, just keeping the  $T$ -matrix invariant leads to another anyonic symmetry, generated by  $e \rightarrow e^7$ .  $h_{e^7} = 1 + \frac{11}{24}$ . This is an allowed symmetry because,  $e^{2\pi i h_{e^7}} = e^{2\pi i h_e}$ . Thus, if we require the full set of anyonic symmetries which preserve the  $T$  matrix, it will be generated by *both* the transformations  $e \rightarrow e^7$  and  $e \rightarrow e^{11}$ . However demanding that the symmetries preserve conformal structure on the edge restricts us to only consider  $e \rightarrow e^{11}$ .

We expect that these can be realized by considering the scaling dimension preserving symmetries of the the set of  $K$  matrices  $\{K' : K' \text{ is stably equivalent to } K\}$ . These will be the subject of future work.

To conclude, we associated the anyon relabeling symmetry (AS) of a general Abelian topological phase to the group of outer automorphisms of the  $K$ -matrix. We presented the AS of the bosonic  $ADE$  Abelian topological states, and discussed thoroughly the  $\mathbb{Z}_2$  symmetry for  $su(3)$  and  $S_3$  triality symmetry for  $so(8)$ . One dimensional gapped interface phases for chiral  $ADE$  states were shown to be naturally classified by AS. A similar method was applied to gapped edge phases for bosonic  $ADE$  fractional quantum spin Hall states, where extra constraints (in addition to the even and odd criterion in Refs. 51 and 52) on the AS and the QP pair condensate were required for TR to be unbroken explicitly or spontaneously. It would be interesting to explore the interplay between AS and TR as well as the compatibility between QP pair condensates and TR in general fermionic phases. We studied topological point defects, each associated to an AS operation, and exhaustively described the fusion behavior of all possible twist defects in the  $ADE$  states. Although not shown explicitly in the letter, the  $F$ -symbols for defect states transformations should take a similar form to certain previously studied exact solvable models in Refs. 11 and 12. These twist defects therefore form a consistent *fusion category* [1], and are powerful enough to construct a measurement-only topological quantum computer [56].

**Note:** During the preparation of the manuscript a recent work of Lu and Fidkowski appeared[55]. Their nice work has similar themes and results to part of our work and both works compliment each other.

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- [1] A. Kitaev, Ann. Phys. **321**, 2 (2006)
  - [2] H. Bombin, Phys. Rev. Lett. **105**, 030403 (2010)
  - [3] H. Bombin, New J. Phys. **13**, 043005 (2011)
  - [4] A. Kitaev and L. Kong, Commun. Math. Phys. **313**, 351 (2012)
  - [5] Y.-Z. You and X.-G. Wen, Phys. Rev. B **86**, 161107(R) (2012)
  - [6] Y.-Z. You, C.-M. Jian, and X.-G. Wen, Phys. Rev. B **87**, 045106 (2013)
  - [7] M. Barkeshli and X.-L. Qi, Phys. Rev. X **2**, 031013 (2012)
  - [8] M. Barkeshli and X.-L. Qi, arXiv:1302.2673(2013)
  - [9] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Phys. Rev. B **87**, 045130 (2013)
  - [10] A. Mesaros, Y. B. Kim, and Y. Ran, arXiv:1305.0214(2013)
  - [11] J. C. Y. Teo, A. Roy, and X. Chen, arXiv:1306.1538(2013)

- [12] J. C. Y. Teo, A. Roy, and X. Chen, arXiv:1308.5984(2013)
- [13] F. D. M. Haldane, Phys. Rev. Lett. **74**, 2090 (1995)
- [14] M. Levin and Z.-C. Gu, Phys. Rev. B **86**, 115109 (2012)
- [15] M. Levin, Phys. Rev. X **3**, 021009 (2013)
- [16] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Phys. Rev. B **88**, 241103(R) (2013)
- [17] M. Barkeshli, C.-M. Jian, and X.-L. Qi, Phys. Rev. B **88**, 235103 (2013)
- [18] J. Cano, M. Cheng, M. Mulligan, C. Nayak, E. Plamadeala, and J. Yard, arXiv:1310.5708(2013)
- [19] Y.-M. Lu and D.-H. Lee, arXiv:1311.1807(2013)
- [20] A. Kitaev, Ann. Phys. **303**, 2 (2003)
- [21] X.-G. Wen, Int. J. Mod. Phys. B **4**, 239 (1990)
- [22] X.-G. Wen, Phys. Rev. B **41**, 12838 (1990)
- [23] X.-G. Wen and A. Zee, Phys. Rev. B **46**, 2290 (1992)
- [24] X.-G. Wen, *Quantum Field Theory of Many Body Systems* (Oxford Univ. Press, Oxford, 2004)
- [25] E. Fradkin, *Field Theories of Condensed Matter Physics*, 2nd ed. (Cambridge University Press, 2013)
- [26] G. S. Boebinger, H. W. Jiang, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. **64**, 1793 (1990)
- [27] Y. W. Suen, J. Jo, M. B. Santos, L. W. Engel, S. W. Hwang, and M. Shayegan, Phys. Rev. B **44**, 5947 (1991)
- [28] J. P. Eisenstein, G. S. Boebinger, L. N. Pfeiffer, K. W. West, and S. He, Phys. Rev. Lett. **68**, 1383 (1992)
- [29] C. Wang and M. Levin, arXiv:1311.0767(2013)
- [30] A. Vishwanath and T. Senthil, Phys. Rev. X **3**, 011016 (2013)
- [31] C. Wang and T. Senthil, Phys. Rev. B **87**, 235122 (2013)
- [32] F. J. Burnell, X. Chen, L. Fidkowski, and A. Vishwanath, arXiv:1302.7072(2013)
- [33] C. Wang, A. C. Potter, and T. Senthil, arXiv:1306.3238(2013)
- [34] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, 1990)
- [35] J. Preskill, *Topological quantum computation* (2004) <http://www.theory.caltech.edu/~preskill/ph219/topological.pdf>
- [36] P. Mathieu and D. Senechal, *Conformal field theory* (New York: Springer, 1997)
- [37] J. Frohlich and E. Thiran, J. Stat. Phys. **76**, 209 (1994)
- [38] J. Frohlich, U. M. Studer, and E. Thiran, J. Stat. Phys. **86**, 821 (1997)
- [39] M. N. Khan, J. C. Y. Teo, and T. L. Hughes, to appear soon
- [40] See online supplementary materials.
- [41] Y.-M. Lu and A. Vishwanath, Phys. Rev. B **86**, 125119 (Sep 2012)
- [42] M. Levin and Z.-C. Gu, arXiv:1304.4569(2013)
- [43] L. Fidkowski, X. Chen, and A. Vishwanath, arXiv:1305.5851(2013)
- [44] E. Verlinde, Nucl. Phys. B **300**, 360 (1988)
- [45] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Vol. 1980 (Springer New York, 1972)
- [46] J. Fuchs, *Affine Lie algebras and quantum groups: An Introduction, with applications in conformal field theory* (Cambridge university press, 1995)
- [47] N. H. Lindner, E. Berg, G. Refael, and A. Stern, Phys. Rev. X **2**, 041002 (2012)
- [48] D. J. Clarke, J. Alicea, and K. Shtengel, Nature Commun. **4**, 1348 (2012)
- [49] M. Cheng, Phys. Rev. B **86**, 195126 (2012)
- [50] A. Vaezi, Phys. Rev. B **87**, 035132 (2013)
- [51] M. Levin and A. Stern, Phys. Rev. Lett. **103**, 196803 (2009)
- [52] M. Levin and A. Stern, Phys. Rev. B **86**, 115131 (2012)
- [53] X.-G. Wen, Phys. Rev. Lett. **90**, 016803 (2003)
- [54] H. Bombin and M. A. Martin-Delgado, Phys. Rev. A **77**, 042322 (Apr 2008)
- [55] Y.-M. L. and L. Fidkowski, arXiv:1311.6481(2013)
- [56] P. Bonderson, M. Freedman, and C. Nayak, Phys. Rev. Lett. **101**, 010501 (2008)
- [57] J. E. Moore and X.-G. Wen, Phys. Rev. B **57**, 10138 (Apr 1998)
- [58] J. E. Moore and X.-G. Wen, Phys. Rev. B **66**, 115305 (Sep 2002)
- [59] X.-G. Wen, Advances in Physics **44**, 405 (1995)



## Supplementary Materials

### Anyon theory of the bosonic Abelian $ADE$ states

Here we summarize two dimensional bosonic Abelian topological states with  $ADE$  chiral Kac-Moody (KM) current algebras at level one along boundary edges. We provide their  $K$ -matrices, quasiparticles (QP) labels, braiding, spin and electric charge of the QPs.

The simplest  $K$ -matrices of the Chern-Simons actions that describe Abelian  $ADE$  topological states are given by the Cartan matrices of the corresponding simply-laced algebras [39].  $A_r = su(r+1)$ , for  $r \geq 2$ , and  $D_r = so(2r)$ , for  $r \geq 4$ , form infinite series of Abelian states, each with a  $K$ -matrix of rank  $r$ .

$$(K_{A_r})_{IJ} = 2\delta_{IJ} - (\delta_{I,J+1} + \delta_{I,J-1}) \quad (20)$$

$$(K_{D_r})_{IJ} = 2\delta_{IJ} - (\delta_{I,J+1} + \delta_{I,J-1}) + (\delta_{I,r}\delta_{J,r-1} + \delta_{I,r-1}\delta_{J,r} - \delta_{I,r}\delta_{J,r-2} - \delta_{I,r-2}\delta_{J,r}). \quad (21)$$

There are three exceptional simply-laced Lie algebra  $K_{E_{r=6,7,8}}$  with  $K$ -matrices

$$K_{E_6} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \quad K_{E_7} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix} \quad (22)$$

$$K_{E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} \quad (23)$$

Since all diagonal entries are 2, any such state is bosonic as all local particles are bosons. The  $E_8$  state does not have topological order as  $\det(K_{E_8}) = 1$  so that the anyon content  $\mathcal{A}_{E_8} = \mathbb{Z}^8 / K_{E_8} \mathbb{Z}^8 = 1$  is trivial and all QPs are mutually local. The  $K$ -matrices (or Cartan matrix) can also be read off from the Dynkin diagrams of the Lie algebras (see Fig. 2 in the main text for  $A_r, D_r, E_6$  and Fig. 5 for  $E_7, E_8$ ). By assigning an enumeration  $I = 1, \dots, r$  of the dots in the Dynkin diagram, the non-zero entries of the  $K$ -matrices are given by  $K_{II} = 2$  and  $K_{IJ} = -1$  if dot  $I$  and  $J$  are connected.

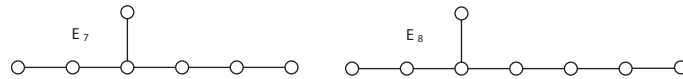


FIG. 5. Dynkin diagrams for the exceptional simply-laced Lie algebras  $E_7, E_8$ .

Even though we are considering purely bosonic states, it is important to note that each Abelian  $ADE$  state can be realized in an electronic system exhibiting a fractional quantum Hall (FQH) state. The  $K$ -matrix of such an electronic  $ADE$  state needs to be modified to include local fermionic electrons. For instance a  $K$ -matrix could take the form of a direct sum  $\mathcal{K} = K_{ADE} \oplus \sigma_z$  [37, 38], where  $\sigma_z$  only introduces local fermions. The electric charge vector would take the form  $\mathbf{t} = (2, \dots, 2, 1, 1)$  so that primitive local bosons are treated as charge  $e^* = 2e$  pairs of electrons. As the additional  $\sigma_z$  does not contribute to the topological order ( $|\det(\sigma_z)| = 1$ ), all results in the main text extends to such electronic FQH states.

Next we describe the anyon lattice vectors corresponding to the QPs in an Abelian  $ADE$  topological state. The bulk boundary correspondence identifies a bulk QP to an edge vertex operator,  $\psi^{\mathbf{a}} \sim e^{i\mathbf{a} \cdot \phi}$ , where  $\mathbf{a}$  is an  $r$ -dimensional

anyon lattice vector in  $\Gamma^* = \mathbb{Z}^r$ . It has charge  $q_{\mathbf{a}}e^*$  for  $q_{\mathbf{a}} = \mathbf{t}^T K^{-1} \mathbf{a}$ , where the charge vector  $\mathbf{t} = (1, 1, \dots, 1)$  describes the external electromagnetic coupling with fundamental charge  $e^*$  for local bosons. As explained in the main text, due to the local boson condensate at zero temperature, the QPs are defined only upto local particles i.e.  $\mathbf{a} \equiv \mathbf{a} + K\mathbb{Z}^r$ . And the fractional electric charge is defined modulo integer multiple of  $e^*$ . Table V lists a particular representation of each set of equivalent anyons  $[\mathbf{a}] = \mathbf{a} + K\mathbb{Z}^r$ .

Algebra	Anyon	Anyon vector
$A_r \equiv su(r+1)$ $r \geq 2$	1 $e^i \ (1 \leq i \leq r)$	$(0, \dots, 0)$ $(0, \dots, \underbrace{1}_{i^{th}}, \dots, 0)$
$D_r(r \text{ odd})$ $r > 4$	1 $e$ $e^2$ $e^3$	$(0, \dots, 0)$ $(0, \dots, 0, 1, 0)$ $(1, \dots, 0)$ $(0, \dots, 0, 1)$
$D_r(r \text{ even})$ $r > 4$	1 $e$ $m$ $\psi$	$(0, \dots, 0)$ $(0, \dots, 0, 1, 0)$ $(0, \dots, 0, 1)$ $(1, \dots, 0)$
$E_6$	1 $e$ $e^2$	$(0, 0, 0, 0, 0, 0)$ $(1, 0, 0, 0, 0, 0)$ $(0, 0, 0, 0, 1, 0)$
$E_7$	1 $e$	$(0, 0, 0, 0, 0, 0, 0)$ $(0, 0, 0, 0, 0, 1, 0)$

TABLE V.  $r$ -dimensional quasiparticle vectors of the  $ADE$  Abelian topological states at level 1.

Algebra	Fractional electric charge
$A_r(r \text{ even})$	All anyons are neutral (or have integral charge)
$A_r(r \text{ odd})$	$e^i \ i \text{ even} \rightarrow$ integral charge $e^i \ i \text{ odd} \rightarrow$ half-integral charge
$D_r(r \bmod 4 = 0, 1)$	All anyons are neutral (or have integral charge)
$D_r(r \bmod 4 = 2)$	$1, \psi \rightarrow$ neutral (or integral charge) $e, m \rightarrow$ half-integral charge
$D_r(r \bmod 4 = 3)$	$1, e^2 \rightarrow$ neutral (or integral charge) $e, e^3 \rightarrow$ half-integral charge
$E_6$	All anyons are neutral (or have integral charge)
$E_7$	$1 \rightarrow$ neutral $e \rightarrow$ half-integral charge

TABLE VI. Fractional electric charges of the quasiparticles in units of fundamental boson charge  $e^*$ .

QP braiding is summarized by the  $S$  matrix, where the phase of

$$S_{\mathbf{a}\mathbf{b}} = \frac{1}{\mathcal{D}} e^{2\pi i \mathbf{a}^T K^{-1} \mathbf{b}} \quad (24)$$

corresponds the braiding phase if the QP  $\psi^{\mathbf{a}}$  is dragged once around  $\psi^{\mathbf{b}}$ . The normalization  $\mathcal{D} = \sqrt{\det(K)}$ , known as the total quantum dimension, is to ensure unitarity of the  $S$  matrix. The spin of a QP is given by the exchange phase

$$\theta_{\mathbf{a}} = e^{2\pi i h_{\mathbf{a}}} = e^{\pi i \mathbf{a}^T K^{-1} \mathbf{a}} \quad (25)$$

from the spin-statistics theorem. These quantities are invariant under the addition of local bosons. The QP's spin and braiding phases are listed in Table VII, and VIII respectively, and are labeled according to the anyon labels in Table V. For instance they verify the triality  $S_3 = Dih_3$  symmetry for  $so(8)$  and the eightfold periodicity  $D_r \rightarrow D_{r+8}$ .

Algebra	$S$ matrix
$A_r$	$S_{e^\mu, e^\nu} = \frac{1}{\sqrt{r+1}} \exp \left[ -2\pi i \frac{\mu\nu}{r+1} \right]$ $0 \leq \mu, \nu \leq r$
$D_r(r \bmod 4 = 0)$	$S_{ee} = S_{mm} = S_{\psi\psi} = 1$ $S_{em} = S_{e\psi} = S_{m\psi} = -1$
$D_r(r \bmod 4 = 1)$	$S_{e^\mu, e^\nu} = \frac{1}{2} \exp \left[ \pi i \frac{\mu\nu}{2} \right] \quad 0 \leq \mu, \nu \leq 3$
$D_r(r \bmod 4 = 2)$	$S_{ee} = S_{mm} = S_{e\psi} = S_{m\psi} = -1$ $S_{em} = S_{\psi, \psi} = 1$
$D_r(r \bmod 4 = 3)$	$S_{e^\mu, e^\nu} = \frac{1}{2} \exp \left[ -\pi i \frac{\mu\nu}{2} \right] \quad 0 \leq \mu, \nu \leq 3$
$E_6$	$S_{e,e} = S_{e^2, e^2} = S_{e, e^2}^* = \frac{1}{\sqrt{3}} \exp \left[ \frac{2\pi i}{3} \right]$
$E_7$	$S_{e,e} = -\frac{1}{\sqrt{2}}$

TABLE VII. Braiding  $S$ -Matrix of anyons

Algebra	Spin
$A_r$	$h_\mu = \frac{\mu}{2} \left( 1 - \frac{\mu}{r+1} \right)$ $0 \leq \mu \leq r$
$D_r(r \text{ odd})$	$h_1 = 0; h_e = h_{e^3} = \frac{r}{8}; h_{e^2} = \frac{1}{2}$
$D_r(r \text{ even})$	$h_1 = 0; h_e = h_m = \frac{r}{8}; h_\psi = \frac{1}{2}$
$E_6$	$h_1 = 0; h_e = h_{e^2} = \frac{2}{3}$
$E_7$	$h_1 = 0; h_e = \frac{3}{4}$

TABLE VIII. Quasiparticle spin.

### Anyonic relabeling symmetry $Outer(K)$

In the main text, we defined the notion of *anyon relabeling symmetry* by the group of outer automorphisms (5). Here we demonstrate this explicitly for the  $A_2 = su(3)$  state. The group of automorphisms  $Aut(K)$  can be identified with the dihedral group  $Dih_6$ , the symmetry group of a hexagon. It is generated by a sixfold “rotation”  $P$  and a twofold “reflection”  $R$ , and has the representation

$$Aut(K_{su(3)}) = \left\langle P = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \middle| P^6 = R^2 = 1, RPR = P^{-1} \right\rangle. \quad (26)$$

We notice that these matrices are isometries with respect to the  $K$ -matrix for  $su(3)$ ,  $PKP^T = RKR^T = K$ . Also,  $P$  and  $R$  act on anyon labels by taking  $e \leftrightarrow e^2$ .  $P$  and  $R$  can be visualized geometrically in Fig. 1 (c) and 1 (d) respectively. On the other hand the matrices  $PR, P^2$  preserve anyon labels up to local particles, and therefore generate the group of inner automorphisms.  $P^2$  geometrically refers to Fig. 1 (a) and Fig. 1 (b) refers to  $RP^3$ .

$$Inner(K_{su(3)}) = \langle PR, P^2 \rangle = \mathbb{Z}_2 \ltimes \mathbb{Z}_3 = S_3. \quad (27)$$

The quotient

$$Outer(K_{su(3)}) = \frac{Aut(K)}{Inner(K)} = \frac{Dih_6}{S_3} = \mathbb{Z}_2 = \{1, \sigma\} \quad (28)$$

describes a mirror symmetry of the  $A_2 = su(3)$  state,  $e \leftrightarrow e^2$ . We further notice that the equivalence class 1 can be represented by any of the elements  $1, P^2, P^4, PR, P^3R, P^5R$ , while the equivalence class  $\sigma$  can be represented by  $R, P, P^3, P^5, P^2R, P^4R$ . However if we further impose the constraint of charge conservation, the symmetry leaves  $\mathbf{t} = (1, 1)$  invariant, and thus we are restricted to the identity matrix for conjugacy class 1 and  $R$  for mirror  $\sigma$ .

For a general  $ADE$  state,  $\text{Outer}(K) = \mathbb{Z}_2$  except for  $D_4 = so(8)$ , where  $\text{Outer}(K_{so(8)}) = S_3$  and is explained in the main text. Given any  $ADE$  state, it is always possible to find a *charge conserving* symmetry matrix which realizes the symmetry. It has been already explicitly written down in the main text for the  $A_2 = su(3)$  and  $D_4 = so(8)$  states. The charge conserving matrices that represent the mirror anyonic symmetries  $\sigma$  in Table II for other simply-laced Lie algebras are listed in Eq. (29). There is no mirror anyonic symmetry for  $E_7$  as its symmetry group is trivial.

$$\sigma_{A_r} = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}_{r \times r}, \quad \sigma_{D_r} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \sigma_x \end{pmatrix}_{r \times r}, \quad \sigma_{E_6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (29)$$

### Stability of gapped interface phase

The set of sine-Gordon coupling terms in (8) introduce a finite energy gap for all of the edge degrees of freedom, and correspond to a particular gapped phase along the interface. By rearranging the bosons  $\phi_I^R = \varphi + (K^{-1})_{IJ}\theta_J$  and  $\phi_I^L = \varphi_I - (K^{-1})_{IJ}\theta_J$ , eq.(8) becomes an ordinary gapped sine-Gordon model

$$\mathcal{L}_M = \frac{1}{\pi} \partial_x \varphi_I \partial_t \theta_I - g_I \cos(2\tilde{\theta}) \quad (30)$$

where  $\langle 2\tilde{\theta}_I \rangle = \langle K_{IJ}(\phi_J^R + M_{J'J}\phi_{J'}^L) \rangle$  is pinned at 0 modulo  $2\pi$  when  $g_I$  is big for all  $I$ .

In the strong coupling limit, the collection of backscattering terms  $\{g_I^{(M)}\}$  with respect to a symmetry  $M$  in Eq. (8) describes a fully gapped interface phase  $\mathcal{L}_M$ . Their scaling dimensions  $\Delta(g_I^{(M)})$  determine the low energy relevance at the fixed point in the renormalization group sense. Similar to conventional Luttinger liquid theory, they depend on the forward scattering Hamiltonian

$$\mathcal{H} = V_{IJ}^{\sigma\sigma'} \partial_x \phi_I^\sigma \partial_x \phi_J^{\sigma'} \quad (31)$$

The backscattering terms  $g_I^{(M)}$  can be simultaneously tuned to be relevant by an appropriate choice of  $V_{IJ}^{\sigma\sigma'}$  [18, 57, 58].

### Time Reversal Invariance(TRI) on the edge for bosonic systems in the presence of gapping terms

#### General formulation

In this subsection we begin with a brief discussion of TRI for bosonic systems, without any reference to the particular form of the  $K$  matrix or charge vector  $\mathbf{t}$ . We follow the discussions in [41, 52].

Let us consider the bulk CS action

$$\mathcal{L}_{\text{bulk}} = \frac{K_{IJ}}{4\pi} \epsilon^{\mu\nu\lambda} \alpha_{I\mu} \partial_\nu \alpha_{J\lambda} - \frac{e^*}{2\pi} t_I \epsilon^{\lambda\mu\nu} A_\lambda \partial_\mu \alpha_{I\nu}, \quad I, J = 1, 2, \dots, N. \quad (32)$$

We impose TRI on the system and study the implications on the bulk and the edge. Here we have assumed that there are  $N$   $U(1)$  gauge fields  $\alpha_I$ , thus the  $K$  matrix is  $N \times N$  and the gauge group is  $U(1)^N$ . The system being bosonic, the diagonal entries of the  $K$  matrix must be even

$$K_{II} = 0 \pmod{2}.$$

Under the action of the anti-unitary time reversal operator  $\mathcal{T}$ , the external electromagnetic gauge field  $A$  transforms as

$$\begin{aligned} A^0 &\rightarrow A^0 \\ A^i &\rightarrow -A^i; \quad i = 1, 2. \end{aligned}$$



$\mathcal{T}$  acts on the internal CS gauge fields  $\alpha_I$  as

$$\alpha_{I\mu} \rightarrow \mp T_{IJ} \alpha_{J\mu} \quad (33)$$

where the  $-$  sign stands for the time index  $\mu = 0$  and the  $+$  sign stands for the spatial indices  $\mu = 1, 2$ . Here,  $T$  is an integer valued  $N \times N$  matrix which has to obey some constraints as outlined below.

Imposing TRI on  $\mathcal{L}_{\text{bulk}}$  and using equation (33) leads to

$$T^T K T = -K \quad (34)$$

$$T \mathbf{t} = \mathbf{t}. \quad (35)$$

For a bosonic system,  $\mathcal{T}^2 = \mathbf{1}$ . Hence,

$$T^2 = \mathbf{1} \quad (36)$$

Next, let us consider the edge of the system in (32). The Lagrangian density is

$$\mathcal{L}_{\text{edge}} = \frac{K_{IJ}}{4\pi} \partial_x \phi_I \partial_t \phi_J + \frac{e^*}{2\pi} \epsilon^{\mu\nu} t_I \partial_\mu \phi_I A_\nu + \text{forward scattering terms}. \quad (37)$$

Remembering that  $\partial_\mu \phi = \alpha_\mu$  [24, 59] and using equation (33), we get

$$\mathcal{T}^{-1} \phi_I \mathcal{T} = T_{IJ} \phi_J + C_I, C_I \in \mathbb{R}.$$

Here  $C_I$  is a constant which will be fixed later from physical considerations. For notational convenience and to align our expressions with previous work, let us replace  $C_I$  by  $\pi (K^{-1})_{IJ} \chi_J$ , i.e.,

$$\mathcal{T}^{-1} \phi_I \mathcal{T} = T_{IJ} \phi_J + \pi (K^{-1})_{IJ} \chi_J, \chi \in \mathbb{R}^N. \quad (38)$$

$\chi$  is often referred to as the time reversal vector.

Physically  $\chi$  determines the action of time reversal  $\mathcal{T}$  on vertex operators  $\psi^{\mathbf{a}} = e^{i\mathbf{a} \cdot \phi}$  on the edge. However, different  $\chi$ 's are not necessarily physically distinct. In fact they might be gauge equivalent to each other.

To understand this we start off by noting that  $\mathcal{L}_{\text{edge}}$  is translationally invariant in  $\phi$ . Thus,  $\phi \rightarrow \phi + \xi$  leaves  $\mathcal{L}_{\text{edge}}$  unchanged. This should come as no suprise as the gauge field  $\alpha_{I\mu}$  is also left unchanged by this redefinition. Indeed,  $\alpha_{I\mu} = \partial_\mu(\phi_I + \xi_I) = \partial_\mu \phi_I$ . But,  $\phi \rightarrow \phi + \xi \implies e^{i\mathbf{a} \cdot \phi} \rightarrow e^{i\mathbf{a} \cdot \phi} e^{i\mathbf{a} \cdot \xi}$ .

Consider the *global*  $U(1)$  gauge transformation on the edge associated with the translational invariance of  $\phi$ ,

$$\tilde{\psi}^{\mathbf{a}} = \psi^{\mathbf{a}} e^{i\mathbf{a} \cdot \xi} \quad \text{where } \psi^{\mathbf{a}} = e^{i\mathbf{a} \cdot \phi}. \quad (39)$$

With this we can see how  $\tilde{\psi}^{\mathbf{a}}$  and  $\psi^{\mathbf{a}}$  transform under TR:

$$\mathcal{T}^{-1} \psi^{\mathbf{a}} \mathcal{T} = \left[ \psi^{(T^T \mathbf{a})} \right]^\dagger e^{-i\pi (K^{-1} \mathbf{a}) \cdot \chi} \quad (40)$$

$$\implies \mathcal{T}^{-1} \tilde{\psi}^{\mathbf{a}} \mathcal{T} = \mathcal{T}^{-1} \psi^{\mathbf{a}} e^{i\mathbf{a} \cdot \xi} \mathcal{T} = \left[ \psi^{(T^T \mathbf{a})} \right]^\dagger e^{-i\pi (K^{-1} \mathbf{a}) \cdot \chi} e^{-i\mathbf{a} \cdot \xi}. \quad (41)$$

We define the time reversal vector  $\tilde{\chi}$  in the new gauge in terms of the action of  $\mathcal{T}$  on  $\tilde{\psi}^{\mathbf{a}}$ , analogous to equation (40).

$$\mathcal{T}^{-1} \tilde{\psi}^{\mathbf{a}} \mathcal{T} = \left[ \tilde{\psi}^{(T^T \mathbf{a})} \right]^\dagger e^{-i\pi (K^{-1} \mathbf{a}) \cdot \tilde{\chi}}. \quad (42)$$

Combining equations (41),(42), we get

$$\tilde{\chi} = \chi + \frac{1}{\pi} K(1 - T) \xi \pmod{2} \quad (43)$$

Thus  $\chi$  and  $\tilde{\chi}$  are gauge equivalent to each other.

The other constraint on  $\chi$  is determined by the action of  $\mathcal{T}$  on local operators  $\psi_{\text{local}}$  on the edge. Local vertex operators are of the form,  $\psi_{\text{local}} = e^{i\lambda^T K \phi}$ ;  $\lambda \in \mathbb{Z}^N$ . Since the system under consideration is bosonic

$$\mathcal{T}^{-2} \psi_{\text{local}} \mathcal{T}^2 = \psi_{\text{local}}.$$

However, we also know that

$$\mathcal{T}^{-2} \psi_{\text{local}} \mathcal{T}^2 = \mathcal{T}^{-2} e^{i\lambda^T K \phi} \mathcal{T}^2 = e^{i\lambda^T K \phi} e^{i\pi \lambda^T \chi} e^{-i\pi (T \lambda)^T \cdot \chi}.$$

Since this must be true for all  $\lambda$ , we find the constraint

$$(1 - T^T) \chi = 0 \pmod{2}. \quad (44)$$

Equations (43),(44) are very important in the definition of  $\chi$  and we will use them in the next section.

### Time reversal at the edge of bosonic fractional quantum spin Hall states

In the main text, we studied the edge of a bosonic fractional quantum spin Hall system with  $K$  matrix

$$K_{2r \times 2r}^{\sigma\sigma'} = \begin{pmatrix} K_{r \times r} & 0 \\ 0 & -K_{r \times r} \end{pmatrix}, \text{ where } \sigma = R, L = \uparrow, \downarrow.$$

The charge vector for this system is  $\mathbf{t}_{2r \times 1}^\sigma = \begin{pmatrix} \mathbf{t}_{r \times 1} \\ \mathbf{t}_{r \times 1} \end{pmatrix}$ .

A suitable value of  $T$  is

$$T_{IJ}^{\sigma\sigma'} = (\sigma_x)^{\sigma\sigma'} \delta_{IJ} = \begin{pmatrix} 0 & \mathbf{1}_{r \times r} \\ \mathbf{1}_{r \times r} & 0 \end{pmatrix} \quad (45)$$

It obeys  $T^2 = \mathbf{1}$ ;  $T^T K^{\sigma\sigma'} T = -K^{\sigma\sigma'}$  and  $T \mathbf{t}^\sigma = \mathbf{t}^\sigma$ .

Now, we need a time reversal vector  $\chi^\sigma = \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix}$  which obeys (44).

$$(\mathbf{I} - T^T) \chi^\sigma = 0 \bmod 2 \implies \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} = 0 \bmod 2 \implies \chi_\uparrow = \chi_\downarrow \bmod 2 \quad (46)$$

The vector  $\chi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  trivially satisfies this condition.

Next, we claim that *any* valid time reversal vector  $\chi^\sigma$  (which satisfies equation (46)) is gauge equivalent to  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .

Using equation (43) an equivalent statement is

$$\exists \begin{pmatrix} \xi_\uparrow \\ \xi_\downarrow \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} + \frac{1}{\pi} \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \xi_\uparrow \\ \xi_\downarrow \end{pmatrix} \bmod 2 \quad \Bigg| \quad \forall \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} : \chi_\uparrow = \chi_\downarrow \bmod 2$$

This reduces to

$$\begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} = -\frac{1}{\pi} \begin{pmatrix} K(\xi_\uparrow - \xi_\downarrow) \\ K(\xi_\uparrow - \xi_\downarrow) \end{pmatrix} \bmod 2$$

A solution to this equation is some  $\begin{pmatrix} \xi_\uparrow \\ \xi_\downarrow \end{pmatrix}$  such that  $\chi_\uparrow = -\frac{1}{\pi} K(\xi_\uparrow - \xi_\downarrow)$ . Since  $\det(K) \neq 0$  such a solution exists.

Furthermore, since  $\chi_\uparrow = \chi_\downarrow \bmod 2$ ,  $\chi_\downarrow = -\frac{1}{\pi} K(\xi_\uparrow - \xi_\downarrow) \bmod 2$  is automatically satisfied.

In conclusion we have shown that for the purposes of bosonic fractional quantum spin hall states we can fix our gauge so that the time reversal vector  $\chi^\sigma = 0$ , since all other choices of  $\chi^\sigma$  are gauge equivalent to it. Thus from now on we can and henceforth we will use the gauge in which

$$\mathcal{T}^{-1} \phi_I^{L/R} \mathcal{T} = \phi_I^{R/L} \quad (47)$$

### Explicit breakdown of TRI on the edge of bosonic FQSH systems for threefold defects

To answer this question we need a microscopic treatment. We begin by expanding the edge Lagrangian density in (7),

$$\mathcal{L}_{\text{edge, bare}} = \frac{1}{4\pi} K_{IJ} \partial_x \phi_I^R \partial_t \phi_J^R - \frac{1}{4\pi} K_{IJ} \partial_x \phi_I^L \partial_t \phi_J^L + \frac{e^*}{2\pi} \epsilon^{\mu\nu} t_I \partial_\mu \phi_I^R A_\nu + \frac{e^*}{2\pi} \epsilon^{\mu\nu} t_I \partial_\mu \phi_I^L A_\nu.$$

$\mathcal{L}_{\text{edge}}$  is obviously TRI. But, now we need to understand what happens when we add the gapping term  $\delta \mathcal{L}_M$  in equation (8); we reproduce it here for convenience

$$\delta \mathcal{L}_M = - \sum_I g_I^{(M)} \cos [K_{IJ} (\phi_J^L + M_{J'J} \phi_{J'}^R)].$$

$\delta\mathcal{L}_M$  breaks TRI, to restore it we add its time reversed partner

$$\delta\mathcal{L}_{TM} = \mathcal{T}^{-1}(\delta\mathcal{L}_M)\mathcal{T} = -\sum_I g_I^{(M)} \cos[K_{IJ}(\phi_J^R + M_{J'J}\phi_{J'}^L)].$$

The full Lagrangian  $\mathcal{L}_{\text{edge,bare}} + \delta\mathcal{L}_M + \delta\mathcal{L}_{TM}$  is time reversal invariant, but we also need to make sure that  $\delta\mathcal{L}_M$  and  $\delta\mathcal{L}_{TM}$  describe the same gapping phase, i.e. both the terms condense the same set of bosonic quasiparticle pairs. This is equivalent to the statement that

$$\begin{aligned} K_{IJ}(\phi_J^R + M_{J'J}\phi_{J'}^L) &= 2\pi(p_1)_I; \quad (p_1)_I \in \mathbb{Z} \\ \implies K_{IJ}(\phi_J^L + M_{J'J}\phi_{J'}^R) &= 2\pi(p_2)_I; \quad (p_2)_I \in \mathbb{Z}. \end{aligned}$$

Next we find the conditions when the above is true (we use vector notation from here on and  $K$  and  $M$  are matrices)

$$\begin{aligned} K(\phi^R + M^T\phi^L) &= 2\pi\mathbf{p}_1 \\ \text{Using } MKM^T = K \implies K(\phi^L + (M^T)^{-1}\phi^L) &= 2\pi M\mathbf{p}_1. \end{aligned} \tag{48}$$

From the main text, drawing upon the correspondence between gapped interface phases and twist defects, this implies that if  $\delta\mathcal{L}_M$  leads to the gapped interface which acts on the anyons labels  $\mathbf{a} \rightarrow M\mathbf{a}$ ,  $\delta\mathcal{L}_{TM}$  leads to  $\mathbf{a} \rightarrow M^{-1}\mathbf{a}$ .

Further, as remarked before two anyon symmetries  $M$  and  $M'$  lead to the same gapped interface if  $M\mathbf{a} = M'\mathbf{a} + K\mathbb{Z}^r \forall \mathbf{a}$ , which is to say  $M = M'$  upto inner automorphisms. In this case we require  $M = M^{-1}$ , thus  $M^2 = 1$  upto inner automorphisms.

Thus if the gapped edge of the bosonic quantum spin hall effect is in the phase  $\mathcal{L}_M$  and preserves TRI it must satisfy  $M^2 = 1$ . We, have already seen the example of the three fold defect  $\rho$  of  $so(8)$ , which acts on the anyon labels by sending  $(e, m, \psi) \rightarrow (m, \psi, e)$  in the main text. It satisfies  $\rho^3 = 1$ .  $\delta\mathcal{L}_{T\rho}$  however traps the defect  $\rho^{-1}$ , by equation (48).  $\rho^{-1} \neq \rho$  and breaks TRI explicitly.

However, even in the case of twofold defects when  $M^2 = 1$ , the expectation value of the condensate and its time reversed partner must be the same to respect TRI. This more stringent condition has already been explored in the main text, it turns out that the condensate in equation (10) breaks TR unless  $\mathbf{m} = -M\mathbf{m}$  is satisfied as proved in equation (12).